Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 4, pp. 829 - 836

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.4.829

REDUCED RECIPROCAL RANDIĆ ENERGY OF A GRAPH

M. K. NATESH, K. N. PRAKASHA, AND M. MANJUNATHA

In the memory of Prof. Chandrashekar Adiga, University of Mysore, India.

ABSTRACT. In this paper, we define reduced reciprocal Randić matrix of a graph G on n vertices. It is denoted by $RRR(G)$ and is defined as an $n \times n$ matrix whose $(i,j)th-$entry is $\sqrt{(d_{v_i}-1)(d_{v_j}-1)}$ if v_i and v_j are adjacent, and 0 otherwise. The reduced reciprocal Randić energy is the sum of absolute values of the eigenvalues of $RRR(G)$. Reduced reciprocal Randić energy of some well-known and much studied graphs are reported. Also, an upper and lower bound for the reduced reciprocal Randić energy of a graph with respect to a vertex subset is presented.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 05C50

KEYWORDS AND PHRASES. Reduced reciprocal Randić index, reduced reciprocal Randić energy.

1. INTRODUCTION

Reduced reciprocal Randić index is one of the vertex–degree-based graph invariant, which was earlier considered in the chemical and/or mathematical literature, and somehow evaded the attention of most mathematical chemists. But in 2014, Gutman et al. [5] brought back this topological index into the main stream of the mathematical chemistry. It is defined as

$$RRR(G) = \sum_{uv \in E(G)} \sqrt{(d_u-1)(d_v-1)},$$

where $E(G)$ is the edge set of G, d_u and d_v represents the degree of the vertices u and v, respectively.

The energy of a graph G is the sum of the absolute values of the eigenvalues of the adjacency matrix of G [4,12]. It has a correlation with the total $pi-$electron energy of a molecule in the quantum chemistry as calculated with the Hückel molecular orbital method. Studies on Randić indices can be found in [5,13,15–17]. In spectral graph theory, different kinds of energy of a graph G have been extensively studied by many researchers and some of them can be found in [3,6–11,14].

Motivated by the reduced reciprocal Randić index, we introduce the reduced reciprocal Randić matrix $RRR(G)$ as $RRR(G) = (r_{ij})_{n \times n}$, where

$$r_{ij} = \begin{cases} \sqrt{(d_{v_i}-1)(d_{v_j}-1)}, & v_iv_j \in E(G) \\ 0, & otherwise. \end{cases}$$

The reduced reciprocal Randić energy is given by

$$RRRE(G) = \sum_{i=1}^{n} |\alpha_i|$$

where α_i are the eigenvalues of reduced reciprocal Randić matrix.

Let χ be a subset of vertex set of G (for example, χ may be a minimum dominating, double dominating set, global dominating set, etc.). Then we define reduced reciprocal Randić matrix as $RRR_\chi(G)$ with respect to the set χ as $RRR_\chi(G) = (r_{ij})_{n\times n}$, where

$$r_{ij} = \begin{cases} \sqrt{(d_{v_i}-1)(d_{v_j}-1)}, & v_iv_j \in E(G) \\ 1, & if\ i = j\ and\ v_i \in \chi \\ 0, & otherwise. \end{cases}$$

The reduced reciprocal Randić energy with respect to the set χ, denoted by $RRRE_\chi(G)$, is the sum of the absolute values of the eigenvalues of the matrix $RRR_\chi(G)$.

In Section 2 of the paper, reduced reciprocal Randić energy of some well-known graphs are computed. In Section 3, an upper and lower bound for the reduced reciprocal Randić energy of a graph with respect to a vertex subset χ is obtained.

2. Reduced reciprocal Randić energy of some standard graphs

In this section, we obtain the reduced reciprocal Randić energy of some standard graphs.

Theorem 2.1. *Reduced reciprocal Randić energy of complete graph K_n is $RRRE(K_n) = 2(n-2)(n-1)$.*

Proof. The degree of a vertex in K_n is n–1. Hence its reduced reciprocal Randić matrix is

$$\begin{bmatrix} 0 & n-2 & n-2 & \dots & n-2 & n-2 \\ n-2 & 0 & n-2 & \dots & n-2 & n-2 \\ n-2 & n-2 & 0 & \dots & n-2 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-2 & n-2 & n-2 & \dots & 0 & n-2 \\ n-2 & n-2 & n-2 & \dots & n-2 & 0 \end{bmatrix}.$$

The spectrum of the above matrix consists of $-(n-2)$ with multiplicity $n-1$ and $(n-2)(n-1)$ with multiplicity 1. Therefore, $RRRE(K_n) = 2(n-2)(n-1)$. □

Definition 2.2. *[1] The crown graph S_n^0 is a graph whose vertex set can be partitioned into two sets $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_n\}$ such that u_iv_j is an edge if and only if $i \neq j$.*

Theorem 2.3. *The reduced reciprocal Randić energy of crown graph S_n^0 is*

$$RRRE(S_n^0) = 4(n-1)(n-2).$$

Proof. The reduced reciprocal Randić matrix of S_n^0 is

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & (n-2) & \dots & (n-2) & (n-2) \\ 0 & 0 & 0 & \dots & 0 & (n-2) & 0 & \dots & (n-2) & (n-2) \\ 0 & 0 & 0 & \dots & 0 & (n-2) & (n-2) & \dots & 0 & (n-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & (n-2) & (n-2) & \dots & (n-2) & 0 \\ 0 & (n-2) & (n-2) & \dots & (n-2) & 0 & 0 & \dots & 0 & 0 \\ (n-2) & 0 & (n-2) & \dots & (n-2) & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-2) & (n-2) & 0 & \dots & (n-2) & 0 & 0 & \dots & 0 & 0 \\ (n-2) & (n-2) & (n-2) & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Characteristic equation is

$(\alpha-(n-2))^{n-1}(\alpha+(n-2))^{n-1}(\alpha+(n-1)(n-2))(\alpha-(n-1)(n-2))=0.$

Therefore the spectrum of $RRR(S_n^0)$ is

$$\begin{pmatrix} (n-2)(n-1) & -(n-2)(n-1) & -(n-2) & (n-2) \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Therefore, $RRRE(S_n^0) = 4(n-1)(n-2)$. □

Theorem 2.4. *The reduced reciprocal Randić energy of a complete bipartite graph $K_{m,n}$ is*

$$2\sqrt{mn}\sqrt{mn-m-n+1}$$

Proof. The reduced reciprocal Randić matrix of complete bipartite graph $K_{m,n}$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & A & A & A & A \\ 0 & 0 & 0 & 0 & \dots & A & A & A & A \\ 0 & 0 & 0 & 0 & \dots & A & A & A & A \\ 0 & 0 & 0 & 0 & \dots & A & A & A & A \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ A & A & A & A & \dots & 0 & 0 & 0 & 0 \\ A & A & A & A & \dots & 0 & 0 & 0 & 0 \\ A & A & A & A & \dots & 0 & 0 & 0 & 0 \\ A & A & A & A & \dots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here $A = \sqrt{mn-m-n+1}$.

Therefore, $Spec_{RRR}(K_{m,n}) =$

$$\begin{pmatrix} \sqrt{mn}\sqrt{mn-m-n+1} & 0 & -\sqrt{mn}\sqrt{mn-m-n+1} \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

Hence, $RRRE(K_{m,n}) = 2\sqrt{mn}\sqrt{mn-m-n+1}$. □

Definition 2.5. *[1] The cocktail party graph, denoted by $K_{n\times 2}$, is a graph with vertex set $V = \cup_{i=1}^{n}\{u_i, v_i\}$ and edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \le i < j \le n\}$.*

Theorem 2.6. *The reduced reciprocal Randić energy of $K_{n\times 2}$ is*

$$RRRE(K_{n\times 2}) = 4(2n-3)(n-1).$$

Proof. The reduced reciprocal Randić matrix is

$$\begin{bmatrix} 0 & 0 & 2n-3 & 2n-3 & \dots & 2n-3 & 2n-3 & 2n-3 & 2n-3 \\ 0 & 0 & 2n-3 & 2n-3 & \dots & 2n-3 & 2n-3 & 2n-3 & 2n-3 \\ 2n-3 & 2n-3 & 0 & 0 & \dots & 2n-3 & 2n-3 & 2n-3 & 2n-3 \\ 2n-3 & 2n-3 & 0 & 0 & \dots & 2n-3 & 2n-3 & 2n-3 & 2n-3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2n-3 & 2n-3 & \dots & 2n-3 & 2n-3 & 0 & 0 & 2n-3 & 2n-3 \\ 2n-3 & 2n-3 & \dots & 2n-3 & 2n-3 & 0 & 0 & 2n-3 & 2n-3 \\ 2n-3 & 2n-3 & \dots & 2n-3 & 2n-3 & 2n-3 & 2n-3 & 0 & 0 \\ 2n-3 & 2n-3 & \dots & 2n-3 & 2n-3 & 2n-3 & 2n-3 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$Spec_{RRR}(K_{n\times 2}) = \begin{pmatrix} 2(2n-3)(n-1) & 0 & -2(2n-3) \\ 1 & n & n-1 \end{pmatrix}.$$

Hence, $RRRE(K_{n\times 2}) = 4(2n-3)(n-1)$. □

3. Properties of reduced reciprocal Randić energy

The following result gives the first three coefficients of the reduced reciprocal Randić characteristic polynomial and can be easily proven using the definition of characteristic polynomial.

Proposition 3.1. *In the reduced reciprocal Randić charecteristic polynomial $\phi_{RRR}(G,\alpha)$, the first three coefficients are* 1, 0 *and* $-\sum_{i=1}^{n}(d_u-1)(d_v-1)$, *respectively.*

Proof. (i) From the definition of the characteristic polynomial, we get $a_0 = 1$.

(ii) The sum of the determinants of all 1×1 principal submatrices is equal to the trace.

$$a_1 = (-1)^1 \cdot \text{trace of } [RRR(G)] = 0.$$

(iii) We have

$$\begin{aligned} (-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - a_{ji}a_{ij} \\ &= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} a_{ji}a_{ij} \\ &= -\sum_{i=1}^{n}(d_u-1)(d_v-1). \end{aligned}$$

□

Proposition 3.2. *We have*

$$\sum_{i=1}^{n} {\alpha_i}^2 = 2\sum_{i=1}^{n}[(d_u-1)(d_v-1)],$$

where α_i represents reduced reciprocal Randić eigenvalues.

Proof. We know that

$$\begin{aligned}\sum_{i=1}^{n}\alpha_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}a_{ji}\\ &= 2\sum_{i<j}a_{ij}^2+\sum_{i=1}^{n}a_{ii}^2\\ &= 2\sum_{i<j}a_{ij}^2\\ &= 2\sum_{i=1}^{n}[(d_u-1)(d_v-1)].\end{aligned}$$

□

Theorem 3.3. *Let G be a graph with n vertices. Then*

$$RRRE_\chi(G)\leq\sqrt{2n\sum_{i=1}^{n}[(d_u-1)(d_v-1)-|\chi|]}.$$

Proof. From Cauchy-Schwartz inequality,

$$\left(\sum_{i=1}^{n}x_iy_i\right)^2\leq\left(\sum_{i=1}^{n}{x_i}^2\right)\left(\sum_{i=1}^{n}{y_i}^2\right).$$

Let $x_i=1$ and $y_i=\alpha_i$. Then

$$\left(\sum_{i=1}^{n}|\alpha_i|\right)^2\leq\left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}|\alpha_i|^2\right)$$

which implies that

$$RRRE_\chi(G)\leq\sqrt{2n\sum_{i=1}^{n}[(d_u-1)(d_v-1)-|\chi|]}$$

□

Theorem 3.4. *Let G be a graph with n vertices. Then*

$$RRRE_\chi(G)\geq\sqrt{|\chi|+2\sum_{i=1}^{n}[\sqrt{(d_u-1)(d_v-1)}]^2+n(n-1)[Det(RRR(G))]^{\frac{2}{n}}}.$$

Proof. By definition,

$$\begin{aligned}(RRRE_{\chi}(G))^2 &= \left(\sum_{i=1}^{n} \mid \alpha_i \mid\right)^2 \\ &= \sum_{i=1}^{n} \mid \alpha_i \mid \sum_{j=1}^{n} \mid \alpha_j \mid \\ &= \left(\sum_{i=1}^{n} \mid \alpha_i \mid^2\right) + \sum_{i\neq j} \mid \alpha_i \mid\mid \alpha_j \mid .\end{aligned}$$

Using arithmetic mean and geometric mean inequality, we have

$$\frac{1}{n(n-1)} \sum_{i\neq j} \mid \alpha_i \mid\mid \alpha_j \mid \geq \left(\prod_{i\neq j} \mid \alpha_i \mid\mid \alpha_j \mid\right)^{\frac{1}{n(n-1)}} .$$

Therefore,

$$\begin{aligned}(RRRE_{\chi}(G))^2 &\geq \sum_{i=1}^{n} \mid \alpha_i \mid^2 + n(n-1)\left(\prod_{i\neq j} \mid \alpha_i \mid\mid \alpha_j \mid\right)^{\frac{1}{n(n-1)}} \\ &\geq \sum_{i=1}^{n} \mid \alpha_i \mid^2 + n(n-1)\left(\prod_{i=1}^{n} \mid \alpha_i \mid^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= |\chi| + 2[\sum_{i=1}^{n} \sqrt{(d_u-1)(d_v-1)}] + n(n-1)[|RRR|]^{\frac{2}{n}}.\end{aligned}$$

$$RRRE_{\chi}(G) \geq \sqrt{|\chi| + 2[\sum_{i=1}^{n} \sqrt{(d_u-1)(d_v-1)}] + n(n-1)[|RRR|]^{\frac{2}{n}}}.$$

□

Definition 3.5. *[12] Let G and H be two graphs. The join $G \vee H$ of G and H is a graph obtained from G and H by joining each vertex of G to every vertex in H.*

Lemma 3.6. *[2] For $i = 1, 2$, let M_i be a normal matrix of order n_i having all its row sums equal to r_i. Suppose $r_i, \theta_{i2}, \theta_{i3}, ..., \theta_{in_i}$ are the eigenvalues of M_i, then for any two constants a and b, the eigenvalues of*

$$M := \begin{bmatrix} M_1 & aJ_{n_1\times n_2} \\ bJ_{n_2\times n_1} & M_2 \end{bmatrix}$$

are θ_{ij} for $i = 1, 2,\ j = 2, 3, ..., n_i$ and the two roots of the quadratic equation $(x - r_1)(x - r_2) - abn_1n_2 = 0$.

Theorem 3.7. *Let G_1 be a r_1-regular graph of order n_1 and let G_2 be a r_2-regular graph of order n_2. Then the spectrum of $RRR(G_1 \vee G_2)$ consists of $(r_1 + n_2 - 1)\lambda_i(G_1)$ and $(r_2 + n_1 - 1)\lambda_j(G_2)$ and the two roots of the* *quadratic equation* $(x-(r_1+n_2-1)r_1)(x-(r_2+n_1-1)r_2)-(r_1+n_2-1)(r_2+n_1-1)n_1n_2$

Proof. Since G_1 and G_2 are regular graphs, the RRR matrix of $G_1 \vee G_2$ can be obtained as follows:

$$\begin{bmatrix} (r_1+n_2-1)A(G_1) & [(r_1+n_2-1)(r_2+n_1-1)]^{1/2}J_{n_1\times n_2} \\ [(r_1+n_2-1)(r_2+n_1-1)]^{1/2}J_{n_2\times n_1} & (r_2+n_1-1)A(G_2) \end{bmatrix}$$

Setting $a=b=[(r_1+n_2-1)(r_2+n_1-1)]^{1/2}$ in Lemma 3.6, we arrive at the desired result. □

References

[1] C. Adiga, A. Bayad, I. Gutman and S. A. Srinivas, *The minimum covering energy of graph*, Kragujevac Journal of Science, 34 (2012), 39–56.

[2] C. Adiga and B.R. Rakshith, *Upper bounds for the extended energy of graphs and some extended equienergetic graphs*, Opuscula Math. 38 (2018), no. 1, 5–13.

[3] H. A. Alfran, R. Rajendra, P. Siva Kota Reddy, R. Kemparaju and Sami H. Altoum, Spectral Analysis of Arithmetic Function Signed Graphs, *Glob. Stoch. Anal.*, 11(3) (2024), 50-59.

[4] I. Gutman, *The energy of a graph*, Ber. Math. Stat. Sekt. Forschungsz. Graz, 103 (1978), 1-22.

[5] I. Gutman, B. Furtula and C. Elphick, *Three new/old vertex-degree-based topological indices.* MATCH Communications in Mathematical and in Computer Chemistry, 72(2014), 617–632.

[6] V. Lokesha, Y. Shanthakumari and P. Siva Kota Reddy, Skew-Zagreb Energy of Directed Graphs, *Proc. Jangjeon Math. Soc.*, 23(4) (2020), 557-568.

[7] K. V. Madhusudhan, P. Siva Kota Reddy and K. R. Rajanna, Randic type Additive connectivity Energy of a Graph, *Vladikavkaz Mathematical Journal*, 21(2) (2019), 18-26.

[8] K. N. Prakasha, P. Siva kota Reddy and I. N. Cangul, Partition Laplacian Energy of a Graph, *Adv. Stud. Contemp. Math., Kyungshang*, **27**(4) (2017), 477-494.

[9] K. N. Prakasha, P. Siva kota Reddy and I. N. Cangul, Minimum Covering Randic energy of a graph, *Kyungpook Math. J.*, **57**(4) (2017), 701-709.

[10] K. N. Prakasha, P. Siva kota Reddy and I. N. Cangul, Sum-Connectivity Energy of Graphs, *Adv. Math. Sci. Appl.*, **28**(1) (2019), 85-98.

[11] R. Rajendra, P. Siva Kota Reddy and R. Kemparaju, Eigenvalues and Energy of Arithmetic Function Graph of a Finite Group *Proc. Jangjeon Math. Soc.*, **27**(1) (2024), 29-34.

[12] H.S. Ramane and H.B. Walikar, *Construction of equienergetic graphs*, MATCH Commun. Math. Comput. Chem. 57 (2007), 203–210.

[13] M. Randić, *On characterization of molecular branching*, J. Am. Chem. Soc., 97 (1975), 6609-6615.

[14] P. Siva Kota Reddy, K. N Prakasha and Ismail Naci CANGÜL, Randić Type Hadi Index of Graphs, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.*, 40(4) (2020), 175-181.

[15] R. Todeschini and V. Consonni, *Handbook of Molecular Descriptors*, Wiley-VCH, Weinheim, (2000), 84-90.

[16] R. Todeschini and V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley-VCH, Weinheim, (2009), 161-172.

[17] B. Zhou and N. Trinajstic, *On a novel connectivity index*, J. Math. Chem., 46 (2009), 1252-1270.

Department of Mathematics, PES College of Arts, Science and Commerce,, Mandya, INDIA
Email address: nateshgfgc123@gmail.com

Department of Mathematics,, Vidyavardhaka College of Engineering, Mysuru, INDIA
Email address: prakashamaths@gmail.com

Department of Mathematics, PES College of Arts, Science and Commerce,, Mandya, INDIA
Email address: mmanjunathapes@gmail.com